Transformation Invariance in the Combinatorial Nullstellensatz and Nowhere-Zero Points of Non-Singular Matrices

Uwe Schauz[1], Thomas Honold[2]

[1] Xi'an Jiaotong-Liverpool University (XJTLU), Suzhou, China
uwe.schauz@xjtlu.edu.cn
[2] ZJU-UIUC Institute - Zhejiang University, Haining, China
honold@zju.edu.cn

Abstract

We describe a kind of transformation invariance in the Quantitative Combinatorial Nullstellensatz. This transformation invariance is frequently used to prove list coloring theorems. We describe its usage in a new short proof of Balandraud and Girard's Theorem about zero-sum subsums. We also use the transformation invariance to study nowhere-zero points of non-singular matrices $A \in \mathbb{F}^{n\times n}$, which are points $x \in \mathbb{F}^n$ such that neither x nor Ax have zero entries. Utilizing the non-singularity of A in an elegant way, we give a new proof of Alon and Tarsi's Theorem about the existence of nowhere-zero points over fields $\mathbb{F}$ that are not prime. Afterwards, with other methods, we extend the scope of Alon, Tarsi and Jaeger's Conjecture from fields to rings. Partially proving this extension, we show that over rings that are not fields, every invertible matrix has a nowhere-zero point. Moreover, over the integers modulo m, non-vanishing determinant suffices to guarantee nowhere-zero points, as we prove for all m that are not a prime power. Finally, we show that the four color problem can be stated as an existence problem for nowhere-zero points over the field with three elements.

Keywords : *Combinatorial Nullstellensatz, Balandraud and Girard's Theorem, Jaeger's Conjecture, Alon and Tarsi's Theorem, Nowhere Zero Point, Four Color Theorem.*

1 Introduction

In this paper [11], we do not study *list coloring problems*. But, to the best of our knowledge, transformation invariance in the Combinatorial Nullstellensatz was first used in that field. List colorings of graphs are proper vertex (or edge) colorings where the available colors on each vertex v (or edge e) are given by an individual list L_v (or L_e) of allowed colors. Proving the existence of such colorings can be very difficult, if it has to be proven for all possible lists of colors of given cardinalities. In this regard, the Combinatorial Nullstellensatz can be a helpful tool, because it is possible to choose a polynomial $P \in \mathbb{F}[X_1, X_2, \dots, X_n]$ such that each non-zero of P is a proper coloring of the graph, and because the Combinatorial Nullstellensatz guarantees the existence of a non-zero $x = (x_1, \dots, x_n)$ of P with $x_1 \in I_1, \dots, x_n \in I_n$, for all possible lists $I_1, I_2, \dots, I_n \subseteq \mathbb{F}$ of given cardinalities (see Theorem 1 and Implication (2)). Numerous authors use and explain this idea, see e.g. [4],[10],[14, p. 22],[9, Lemma 1.9]. In these works, however, what we call *transformation invariance* is not explicitly stated. It is not stated in a theorem that one can draw conclusions from the non-zeros of a polynomial P on one domain $\tilde{I}_1 \times \tilde{I}_2 \times \cdots \times \tilde{I}_n$ to the non-zeros on another domain $I_1 \times I_2 \times \cdots \times I_n$. Transformation invariance is implicitly used in some of these papers. For example, in the paper [13] about the list chromatic index of K_8 and K_{10}, Landon Rabern implicitly used transformation invariance. In comparison to that, the list chromatic index of K_6 was calculated without considering that aspect of the Combinatorial Nullstellensatz in [8] by David Cariolaro et al. It seems that using transformation invariance is of advantage when it comes to algorithmic calculations in this kind of problem. But, transformation invariance can also be important in more theoretical approaches, like in the latest attempt to prove the List Edge Coloring Conjecture for complete graphs, in [15]. Transformation invariance, however, was not explicitly stated there either, not in one ready-to-use theorem. We think that stating transformation invariance explicitly might clarify the approach, and that this aspect of the Combinatorial Nullstellensatz is important enough to be explicitly stated in a corollary to the Nullstellensatz. We did that the first time in [16], in connection with list colorings. In this paper, we explain the usage of the described transformation invariance in two other fields, *zero-sum subsums* and *nowhere-zero points*. Beyond that, leaving transformation invariance behind, we further extend the theory of nowhere-zero points to rings and the field $\mathbb{F}_3$:

In Section 3, we study (over the integers modulo a prime p) a problem about zero-sum subsums of given sums $\sum_{j=1}^{n} a_j$. These zero-sum subsums may be regarded as the 0–1 solutions in $\mathbb{Z}/p\mathbb{Z}$ of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\,. \tag{1}$$

The general topic of zero-sum subsums is a broad and well established research field in additive combinatorics and combinatorial algebra. Usually, the main focus is the existence of (non-trivial) zero-sum subsums under different presumptions. In [5], however, Balandraud and Girard asked whether Equation (1) is characterized by its 0–1 solutions, i.e. by the set of all $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ with $a_1x_1 + \cdots + a_nx_n = 0$. They found that if $n > p$ and all coefficients a_j are non-zero, then the coefficients a_j are uniquely determined, up to a constant factor. In other words, in that case, the hyperplane with defining equation $a_1x_1 + \cdots + a_nx_n = 0$ possesses a basis made of 0–1 vectors. This is the Balandraud-Girard Theorem, for which we give a new short proof based on transformation invariance. Other corollaries of the Combinatorial Nullstellensatz, such as the Permanent Lemma [1, Lemma 8.1], can also be used, instead of transformation invariance. Some experts might already be aware of such a short proof, but we are not aware of any published version. The original treatment in [5], whilst also using the Combinatorial Nullstellensatz, is much more complicated. The aim there is to also classify the exceptions to the theorem in the cases $n = p$ and $n = p - 1$.

In Section 4, another interesting question about hyperplanes and bases is examined. It was raised by Jaeger in the form of the following conjecture (originally stated for $q = 5$ only):

Conjecture 1 (Jaeger's Conjecture [12])
To every two bases B_1 and B_2 of an n-dimensional vector space, over the finite field $\mathbb{F}_q$ with $q > 3$ elements, there exists a hyperplane H that is disjoint from B_1 and B_2.

Alon and Tarsi formulated this conjecture as follows, where a vector $x \in \mathbb{F}_q^n$ is called a *nowhere-zero point* of a matrix $A \in \mathbb{F}_q^{n\times n}$ if neither x nor Ax have a zero entry:

Conjecture 2 (Alon and Tarsi's Reformulation [2])
Every non-singular $n \times n$ matrix A, over the finite field $\mathbb{F}_q$ with $q > 3$ elements, has a nowhere-zero point $x \in \mathbb{F}_q^n$.

This conjecture is trivial if $q - 1 > n$ (as well as over infinite fields). In fact, if the first j coordinates of Ax are already non-zero, one can alter x_{j+1} till the first $j+1$ components of Ax differ from zero. Here, one only needs that, after suitable row permutations, the elements on the main diagonal of A are all non-zero. Further partial results, in the case where q is relatively big compared to n, can be found in [3]. Using the Combinatorial Nullstellensatz, however, the conjecture can completely be proven for certain q, with no restriction on n. Alon and Tarsi showed in [2] that the conjecture holds for all proper prime powers q (which means that it holds over every non-prime field). Other proofs of this result can be found in [6] and [7]. Utilizing the assumed non-singularity in a new elegant way based on transformation invariance, we give yet another proof.

In Section 5, we extend the scope of Alon, Tarsi, and Jaeger's Conjecture from fields to rings. We show that over rings that are not fields, every invertible matrix A has a nowhere-zero point. This also holds over non-commutative rings. Over commutative rings, however, it can be strengthened. There, the invertibility of A is equivalent to the invertibility of the determinant $\det(A)$, and one may wonder if the weaker presumption $\det(A) \neq 0$ is enough, over commutative rings. We make the following conjecture:

Conjecture 3 *Over commutative rings R with more than 3 elements, every $n \times n$ matrix A with $\det(A) \neq 0$ has a nowhere-zero point.*

We prove this conjecture for rings with no minimal ideal or more than one minimal ideal. In particular, we show that it holds true over the ring $\mathbb{Z}/m\mathbb{Z}$, whenever m is not a prime power.

Finally, in Section 6 and Theorem 2, we turn to nowhere-zero points over the field $\mathbb{F}_3$ with three elements, and show that the four color problem can be stated as a nowhere-zero point problem over $\mathbb{F}_3$. Of course, there are non-singular matrices over $\mathbb{F}_3$ without nowhere-zero points, but if one could classify those exceptions, it would possible help with the four color problem.

2 References: figures, equations and theorems

Theorem 1 (Quantitative Combinatorial Nullstellensatz [14])
Let $I_1, I_2, \ldots, I_n$ be finite non-empty subsets of a field $\mathbb{F}$, set $I := I_1 \times I_2 \times \cdots \times I_n$ and define $d := (d_1, d_2, \ldots, d_n)$ by $d_j := |I_j| - 1$. For polynomials $P = \sum_{\delta \in \mathbb{N}^n} P_\delta X^\delta \in \mathbb{F}[X_1, \ldots, X_n]$ of total degree $\deg(P) \leq d_1 + d_2 + \cdots + d_n$, we have

$$P_d = \sum_{x \in I} N_I(x)^{-1} P(x) ,$$

where $N_I(x_1, \ldots, x_n) := \prod_j \prod_{\xi \in I_j \setminus \{x_j\}} (x_j - \xi) \neq 0$.

This theorem implies Alon's classical Combinatorial Nullstellensatz [1], i.e. the implication

$$P_d \neq 0 \implies \exists x \in I : P(x) \neq 0 , \tag{2}$$

because a sum can only be non-zero if at least one summand is non-zero. This and several other interesting corollaries of the coefficient formula can be found in [14], as well. Somewhat newer is the following transformation invariance, which follows by applying the theorem twice and considering the coefficient P_d as an intermediate step:

Corollary 1 (Transformation Invariance [16]) *For $j = 1, 2, \ldots, n$, let I_j and $\tilde{I}_j$ be finite non-empty subsets of a field $\mathbb{F}$ with $|I_j| = |\tilde{I}_j|$. Let N_I and $N_{\tilde{I}}$ be the corresponding coefficient functions over the cartesian products I and $\tilde{I}$ of these sets. Assume that $P, \tilde{P} \in \mathbb{F}[X_1, \ldots, X_n]$ are polynomials of total degree at most $|I_1| + |I_2| + \cdots + |I_n| - n$. If $\tilde{P}$ and P have the same homogenous component of maximal degree, or if at least $\tilde{P}_d = P_d$, then*

$$\sum_{x \in \tilde{I}} N_{\tilde{I}}(x)^{-1} \tilde{P}(x) = \sum_{x \in I} N_I(x)^{-1} P(x)$$

and, in particular,

$$\sum_{x \in \tilde{I}} N_{\tilde{I}}(x)^{-1} \tilde{P}(x) \neq 0 \implies \exists x \in I : P(x) \neq 0 .$$

Theorem 2 *The four color theorem is equivalent to the existence of nowhere-zero points of all matrices A that can be obtained in the described way, from two planar binary trees that are glued together, as in Fig. 1, to form a 3-regular planar graph.*

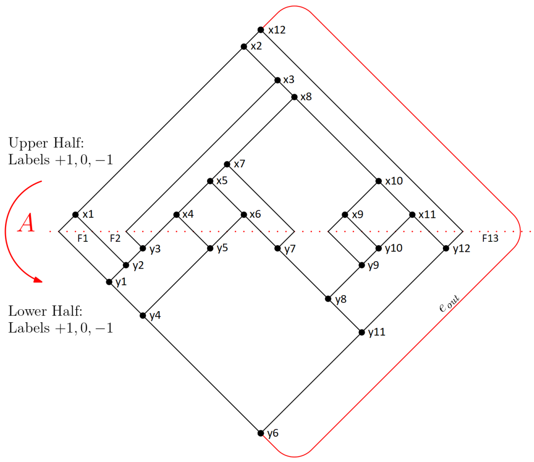

FIG. 1: A 3-regular planar graph with Hamiltonian cycle in its dual graph (dotted line)

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